Embedding of Semisimple Modules and Complemented Modular Lattices

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Abstract. In this work, we present a description of the category of semisimple *R*-modules together with a related category formed by their complemented modular lattices. We also investigate a correspondence that yields an isomorphism between these two categorical structures. Both categories are treated within the framework of normal categories.

***Index Terms*: Semisimple R-modules, Abelian normal category, Complemented modular lattice, Normal cone, Normal mapping.**

I. INTRODUCTION

K.S.S.Nambooripad developed the concept of normal categories as a framework for analysing the structure of regular semigroups (cf.[11]). In such categories, every morphism possesses a normal factorisation, sub objects are well defined, and the set of all normal cones forms a regular semigroup. The notion of an abelian category, due to S. Mac Lane, provides an additional structural framework. In this work, we show that the category of semisimple *R*-modules forms an abelian normal category, denoted by *R-ssmod*, and we also describe a corresponding normal category formed by complemented modular lattices. Since the lattice of submodules of any semisimple *R*-module is itself a complemented modular lattice, and by considering the category whose objects are such lattices with morphisms given by normal mappings, we show that *R-ssmod* is isomorphic to this category of complemented modular lattices.

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II. PRELIMINARIES

We use the standard terminology of category theory through-out. For definitions or results that are not stated explicitly here, the reader may consult the classical references of S. Mac Lane (cf.[8]) and K.S.S.Nambooripad (cf.[11]). In this paper, R denotes a commutative ring with identity, and all categories under consideration are assumed to be small. A functor will be understood as a morphism between cat-egories.

A functor is a morphism of categories. Let C and D be two cate-gories. A functor F: C $\rightarrow$ D is an *embedding(faithful)* when to every pair c,c' of objects of C and to every pair f_1, f_2 : $c \rightarrow c'$ of parallel arrows of C the equality $Ff_1 = Ff_2 : Fc \rightarrow Fc'$ implies $f_1 = f_2$. A functor F : C $\rightarrow$ D is an isomorphism of categories when there is a functor G : D $\rightarrow$ C such that $FG = I_C$ and $GF = I_D$.

Definition 1. *(cf.*[8]*) An abelian category is a category satisfying:*

(1) *each hom-set is an additive abelian group,*
(2) *composition of arrows is bilinear relative to this addition,*
(3) C *has a zero object O,*
(4) C *has binary products and coproducts,*
(5) *every arrow in* C *has a kernel and a cokernel,*
(6) *every monic arrow is a kernel, and every epi a cokernel,*
(7) *zero morphism in each morphism set is the zero of the addition.*

Example 1. *R-mod - Category of R-modules with morphisms R-module homomorphisms is an abelian category.*

A *preorder* P is a category such that, for any $p, p' \in v$P; the hom-set P(p, p') contains at most one morphism. In this case, the relation $\subseteq$ on the class vP defined by $p \subseteq p' \Leftrightarrow$ P$(p, p') \neq \phi$ is a quasi-order on vP . In a preorder, two objects p and p' are isomorphic if and only if P$(p, p') \neq \phi \neq$ P(p', p). Therefore $p \subseteq p'$ is a partial order if and only if P does not contain any nontrivial isomorphisms. Equivalently, the only isomorphisms of P are identity morphisms, and in this case P is said to be a *strict preorder.*

Let C be a category and P be a subcategory of C. Then (C, P) is called a *category with subobjects* if the following hold:

P is a strict preorder with vP = vC.
(1) Every $f \in$ P is a monomorphism in C.
(2) If $f, g \in$ P and if $f = hg$ for some $h \in$ C, then $h \in$ P.

In a category with subobjects, if f: $a \to b$ is a morphism in P, then f is an *inclusion* and is denoted by $j(a, b)$. If there exists a morphism e: $b \to a$ such that $j(a, b)\, e = I_a$, then e is called a *retraction* from $b \to a$, denoted $e(b, a)$. Whenever such a retraction exists, the inclusion $j(a, b) : a \to b$ splits.

A morphism $f \in$ C, where C be a category with subobjects has factorization if $f = p \cdot m$ where p is an epimorphism and m is an embedding. A category C is said to have the *factorisation property* if every morphism of C admits such a decomposotion. In this case, each morphism f in C has atleast one canonical factorisation of the form $f = qj$, where q is an epimorphism and j is an inclusion. A *normal fac-torisation* of a morphism f in C is a factorisation of the form $f = euj$

where e is a retraction, u is an isomorphism and j is an inclusion.

A morphism f in a category with subobjects is said to have an *image* if it has a *canonical(epi-mono) factorization* $f = xj$, with x epi and j an inclusion with the property that for any other canonical fac-torisation $f = yj'$, there exists an inclusion j" such that $y = xj''$([11]).

A category is said to have *images* if every morphism in C has an image.

In this case, the codomain of x is said to be the *image* of f. When the morphism f has an image, we denote the unique canonical factorisation of f by $f = f^o j_f$, where f^o is the *unique epimorphic component* and j_f is the inclusion of f.

Let C be a category with subobjects, images, every morphism in C has normal factorisations in which the inclusion splits and $d \in v$C. A cone from vC to the vertex d is a map $\gamma : v\mathrm{C} \to d$ such that

(1) $\gamma(c) \in \mathrm{C}(c, d)$ for all $c \in v$C.

(2) If $c' \subseteq c$ then $j(c', c)\gamma(c) = \gamma(c')$.

The cone γ is called a *normal cone* if there exists an $a \in v$C such that $\gamma(a)$ is an isomorphism. The vertex d of the cone γ is usually denoted as c_γ. The set of all normal cones in C is denoted by TC.

Definition 2. *A normal category is a pair* (C, P) *satisfying the follow-ing:*

(1) (C, P) *is a category with subobjects*

(2) *every inclusion in C splits*

(3) *Any morphism in C has a normal factorization*

(4) *for each* $a \in v$C *there is a normal cone* γ *with vertex* a *and* $\gamma(a) = I_a$.

If a normal category is also abelian, it is referred to as an *abelian normal category*.

An *ideal* of a complemented modular lattice L is a sublattice I of L such that $x \leq y \in I$ implies $x \in I$. The *principal ideal* $L(x)$ of L generated by $x \in L$ is L(x) = $\{y \in L | y \leq x\}$, which is simply the interval $[0, x]$.

Definition 3. *(cf.*[12]*) Let* L *and* L' *be two complemented modular lattices with bottom and top* *elements* $0, 1$ *in* L *and* $0', 1'$ *in* L'. *A map* $f : L \to L'$ *is said to be a normal mapping if it is order preserving,* $imf = [0', a]$ *for some* $a \in L'$ *and* $\forall x \in L$, $\exists z \leq x$ *such that* $f|_{[0,z]}$ *is an isomorphism from* $[0, z]$ *onto* $[0', f(x)]$.

Definition 4. *A semisimple R-module is an R-module that can be decomposed as a direct sum of simple submodules. A simple module is a non-zero module that has no proper non-trivial submodules.*

Shur's Lemma: Any nonzero homomorphism between simple R-modules is an isomorphism.
Proposition 1. *The submodules of a semisimple R module M form a complemented modular lattice L(M).*

Proof. The submodules of a semisimple module M form a lattice under the partial order of inclusion. The trivial submodules $\{0\}$ and M are the bottom and top elements of L (M). The greatest lower bound is the intersection of submodules and the least upper bound is the sum of submodules. Every submodule W of M has a complement W' such that $M = W \oplus W'$. For $A, B, C \in S(M)$, we have $A \subseteq C \Rightarrow A + (B \cap C) = (A + B) \cap C$. Hence $L(M) = L(S(M), \cap, +, \{0\}, M)$ is a complemented modular lattice.

III. CATEGORY OF SEMISIMPLE MODULES AND COMPLEMENTED MODULAR LATTICES

Theorem 1. *The category of semisimple R-modules with morphisms R-module homomorphisms is an abelian normal category R-ssmod.*
Proof. Consider the category *R-ssmod*, whose objects are semisimple R- modules and whose morphisms are R-module homomorphisms. Since it is a full subcategory of the abelian category R-mod, it is itself abelian. To show that *R-ssmod* is also normal, we proceed as follows.
The category is small, and inclusions of submodules serve as the designated class of subobjects; thus subobjects correspond exactly to submodules. In semisimple modules, every monomorphism is an embedding, so *R-ssmod* indeed has a well-defined notion of subobjects.

If $M \subseteq N$ and $\phi : M \to N$, an inclusion then N decomposes as $N = M \oplus M'$. Define a map $\phi' : N \to M$ as for each $x \in N$,

$$\phi'(x) = \begin{cases} x, if\ x \in M \\ 0, if\ x \in M' \end{cases}$$

Then $\phi \cdot \phi' = I_M$, showing every inclusion splits. Let M and N be semisimple R-modules and $\phi : M \to N$ be an R-module homomorphism. Let K be the kernel of ϕ which is a submodule of M. Then $M = K \oplus K^c$. By first isomorphism theorem, ϕ admits a normal factorization quj where $q : M \to K^c$, retraction; $u : K^c \to im\phi$, iso-morphism and $j : im\phi \to N$, inclusion. Let N_i, $i = 1, 2, ...n$ be semisimple R-modules. Normal cone γ :vR-*ssmod* →R-*ssmod* is defined such that

(1) for each N_i, there is a $\gamma(N_i) : N_i \longrightarrow N_m$

(2) whenever $N_i \subseteq N_k$, $j(N_i, N_k) \cdot \gamma(N_k) = \gamma(N_i)$ where $j(N_i, N_k)$ is an inclusion from N_i to N_k and

(3) there exists at least one $N_l \in \nu$R-*ssmod* such that $\gamma(N_l) : N_l \longrightarrow$ N_m is an isomorphism.

This collection of morphisms $\{\gamma(N_i) : N_i \in \nu\text{R-}ssmod\}$ is a normal cone with vertex N_m. We now show that every object is the vertex of an idempotent normal cone. Let $f : M \rightarrow N_m$ be an R-module homomorphism such that $f(x) = x$, for all $x \in N_m$ and for any $N_i \subseteq M$, define $\gamma(N_i) = f|_N : N_i \longrightarrow N_m$. Then $\gamma(N_m) = I_N$. Thus γ is an idempotent normal cone with vertex $N_m \in \nu$R-*ssmod*. Hence *R-ssmod* satisfies all the axioms of a normal category, completing the proof.

We now construct a normal category, denoted CM*odLat*, whose objects are complemented modular lattices and whose morphisms are normal mappings.

Theorem 2. *The category with complemented modular lattices as objects and morphisms are normal mappings, forms a normal category* ‘CM*odLat*′.

Proof. In ‘CM*odLat*′ objects are complemented modular lattices and morphisms are normal mappings between them. For a normal mapping, $f : L \longrightarrow L'$ the lattice L serves as the domain and L' as the codomain. Composition of two normal mappings is again a normal mapping and the identity map $I_L : L \rightarrow L$ is also normal. Since associativity for composition is inherited from funtion composition and $f \cdot I_L = f = {}' \cdot f$, the structure indeed forms a category.

The ideals of the lattices, $[0, a]$ act as subobjects. If $f : L \subseteq L'$ where $L = [0, a]$ and $L' = [0, b]$ where $a \leq b$; then there exists a map $e : L' \longrightarrow L$ such that $e(c) = \begin{cases} c, \text{if } c \in L \\ 0, \text{if } c \in [a, b] \end{cases}$

showing that every inclusion splits. Moreover, by the third condition of normal mapping, for each $x \in L$, there exists some $b \leq x$ such that the restriction $f|[0, b] : [0, b] \rightarrow [0, f(x)]$ is an isomorphism. Hence every morphism $f : L \rightarrow L'$ admits a normal factorization $f = quj$ where $q : L \rightarrow [0, b]$ is a retraction, $u : [0, b] \rightarrow [0, f(x)]$ is an isomorphism and $j : [0, f(x)] \rightarrow L'$ is an inclusion. A normal cone with vertex L is a map $\gamma : \nu$CM*odLat* $\rightarrow L$ such that

(1) $\gamma(L_1) \in hom(L_1, L)$

(2) if $L_1 \subseteq L_2$, then $j(L_1, L_2) \cdot \gamma(L_2) = \gamma(L_1)$

(3) there exists atleast one L_3 such that $\gamma(L_3)$ is an isomorphism.

There is also an idempotent normal cone with vertex L obtained by taking $\gamma(L) = I_L$. Hence ‘CM*odLat*′ is a normal category.

3.1. SUBMODULES AND COMPLEMENTED MODULAR LATTICE.

We now focus on the complemented modular lattices that arise from the submodule structure of semisimple R-modules and ex-amine the associated normal mappings.

By Proposition 1, the family of submodules of a semisimple R-module M forms a complemented modular lattice, denoted $L(M)$. Let $\theta : M \to N$ be an R-module homomorphism between semisimple R-modules M and N, then by Shur's lemma θ is an isomorphism. For *each submodule Mi* $\in L(M)$, *define* $f\theta(Mi) = \theta(Mi)$.

This gives a map $f_\theta : L(M) \to L(N)$, *which is a normal mapping between the* complemented modular lattices $L(M)$ and $L(N)$. Indeed if $M_1 \leq M_2$, then $\theta(M_1) \leq \theta(M_2) \to f_\theta(M_1) \leq f_\theta(M_2)$, so f_θ preserves the order. The image of f_θ is the ideal generated by N. Moreover, for every M_i in $L(M)$, there exists $M_j \leq M_i$ such that the restriction $f_\theta|_{[0,M]} : [0, M_j] \to [o, f_\theta(M_i)]$ is an isomorphism. Let us call this normal mapping f_θ a 'linear normal mapping' as it coincides with R-module homomorphisms.

Define $hom(L(M), L(N)) = \{f_\theta | \theta \in hom_R(M, N)\}$, where $f_\theta(N) = \theta(N)$ and each f_θ is order preserving, join preserving and a linear nor-mal mapping. It follows that the complemented modular lattices ob-tained from the submodules of semisimple R-modules together with these linear normal mappings, form a normal category, which we denote by C*modL*. This category is naturally a subcategory of CM*odLat*.

Proposition 2. *The sum of two linear normal mappings in* C*modL is again a linear normal mapping.*

Proof. Let M_1 and M_2 be semisimple R-modules and let $\theta, \psi \in hom_R(M_1, M_2)$, define $(f_\theta + f_\psi)(N) = f_{\theta+\psi}(N) = (\theta + \psi)(N) = \theta(N) + \psi(N)$, for all $N \in L(M_1)$. Since $\theta+\psi \in hom_R(M_1, M_2)$, $f_{\theta+\psi} \in hom(L(M_1), L(M_2))$. For $N_1 \subseteq N_2$, then $\theta(N_1) \subseteq \theta(N_2)$ and $\psi(N_1) \subseteq \psi(N_2)$. Hence,

$$f_{\theta+\psi}(N_1) = \theta(N_1) + \psi(N_1) \subseteq \theta(N_2) + \psi(N_2) = f_{\theta+\psi}(N_2),$$

so $f_{\theta+\psi}$ is order-preserving.

Since $L(M_1)$ is the lattice of submodules of M_1, and for $N_1, N_2 \in L(M_1)$ the join $N_1 \vee N_2 = N_1 + N_2$. Using linearity,

$$\begin{aligned} f_{\theta+\psi}(N_1 + N_2) &= (\theta + \psi)(N_1 + N_2) \\ &= \theta(N_1 + N_2) + \psi(N_1 + N_2) \\ &= \theta(N_1) + \theta(N_2) + \psi(N_1) + \psi(N_2) \\ &= f_{\theta+\psi}(N_1) + f_{\theta+\psi}(N_2). \end{aligned}$$

Thus, $f_{\theta+\psi}$ preserves joins.

Since M_1 is semisimple, the image of $\theta + \psi$ is a submodule of M_2, $\mathrm{Im}(f_{\theta+\psi}) = \{(\theta + \psi)(N$ and:
$) \mid N \subseteq M_1\} \subseteq [0, (\theta + \psi)(M_1)]$.

Also, since $f_{\theta+\psi}(M_1) = (\theta + \psi)(M_1)$, we conclude:

$\mathrm{Im}(f_{\theta+\psi}) = [0, f_{\theta+\psi}(M_1)]$,

so the image is a principal ideal. Fix $x \in L(M_1)$. Since f_θ and f_ψ are linear normal mappings, there exist submodules $z_\theta, z_\psi \leq x$ such that $f_\theta|_{[0,z_\theta]} : [0, z_\theta] \cong [0, f_\theta(x)]$, $f_\psi|_{[0,z_\psi]} : [0, z_\psi] \cong [0, f_\psi(x)]$

are lattice isomorphisms. Let $z = z_\theta \cap z_\psi \subseteq x$. Then:

$f_{\theta+\psi}(z) = \theta(z) + \psi(z)$.

Since all submodules of semisimple modules are direct summands, the restrictions of θ, ψ and $\theta+\psi$ to z are injective onto their images. Hence,

$f_{\vartheta+\psi}|_{[0,z]} : [0, z] \xrightarrow{\sim} [0, f_{\theta+\psi}(x)]$

is a lattice isomorphism. Hence $f_{\theta+\psi}$ is again a linear normal mapping in C*modL*.

Proposition 3. *Let M_1 and M_2 be semisimple R-modules, $L(M_1)$ and $L(M_2)$ be complemented modular lattices obtained from submodules of M_1 and M_2 respectively. Then $hom(L(M_1), L(M_2))$ is an additive abelian group.*

Proof. Since endomorphism ring of R-modules is an R-module, $hom(M_1, M_2)$ is an R-module and hence an additive abelian group. Each module homomorphism $\theta \in hom_R(M_1, M_2)$ induces a correspond-ing lattice homomorphism $f_\theta : L(M_1) \to L(M_2)$. Since addition of module homomorphisms corresponds to addition of their induced lat tice maps, the collection $hom(L(M_1), L(M_2))$ inherits an abelian group structure from $hom_R(M_1, M_2)$. Thus $hom(L(M_1), L(M_2))$ forms an ad-ditive abelian group. □

Theorem 3. *The normal category* C*modL is an abelian category.*

Proof. We consider the normal category C*modL* whose objects are com-plemented modular lattices $L(M)$ arising from semisimple R-modules M and whose morphisms are linear normal mappings f_θ induced by R-module homomorphisms $\theta \in hom_R(M_1, M_2)$. For each pair of objects $L(M_1)$, $L(M_2)$, the hom-set $hom(L(M_1), L(M_2))$ is an abelian group under pointwise join, defined by

$$(f_\theta + f_\psi)(N) = \theta(N) + \psi(N), \quad \text{for all } N \in L(M_1),$$

where $\theta, \psi \in hom_R(M_1, M_2)$. The identity element of this group comes from the zero R-module homomorphism. Biproducts in this category correspond to direct sums of modules: $L(M) \oplus L(N) = L(M \oplus N)$, since $M \oplus N$ is semisimple if M and N are. The lattice $L(0) = \{0\}$, corresponding to the zero module, serves as both an initial and terminal ob-ject, and thus is a zero object in the category.

Let $f_\theta : L(M_1) \to L(M_2)$ be a morphism induced by $\theta \in hom_R(M_1, M_2)$. The kernel of f_θ corresponds to the lattice $L(\ker\theta)$, and the cokernel corresponds to $L(\mathrm{coker}(\theta))$, since submodule structure is preserved under these con-structions when M_1 and M_2 are semisimple. The normal category C*modL* satisfies the axioms of an abelian category as every morphism has a kernel and cokernel, every monomorphism is the kernel of its cokernel and every epimorphism is the cokernel of its kernel. Hence, under the identification of morphisms with R-module homomorphisms, C$modL$ is abelian.

Theorem 4. *The abelian normal categories R-ssmod(semisimple R-modules and R-module homomorphisms) and* C*modL(complemented mod-ular lattices of submodules of semisimple R-modules with linear normal mappings) are isomorphic.*

Proof. Let F : R-$ssmod$ $\rightarrow$ C$modL$ be a functor mapping semisimple R-module M to the complemented modular lattice $L(M)$. If $M_1 \leq M$ then F maps M_1 to the sublattice $L(N)$. If ϕ is an R-module homomorphism between two semisimple R-modules M and N, then $F(\phi) : L(M) \rightarrow L(N)$ is a linear normal mapping between two complemented modular lattices $L(M)$ and $L(N)$. Since submodules of semisimple modules are direct summands, each $L(M)$ is a comple-mented modular lattice, and every linear normal map is induced by a module homomorphism. This defines a faithful and full functor from R-$ssmod$ $\rightarrow$ C$modL$. Dually define a functor G : C$modL$ $\rightarrow$R-$ssmod$ by sending each lattice $L(M)$ to the underlying module M, and each linear normal mapping to the corresponding module homomorphism. Since morphisms in CmodL are defined to arise from module homo-morphisms, this defines a functor. Hence $F \cdot G \cong I_{R-ssmod}$ and $G \cdot F \cong I_{CmodL}$.

IV. CONCLUDING REMARKS

In this paper we establish a categorical equivalence between the category R-$ssmod$ of semisimple R-modules and the category C$modL$ of complemented modular lattices of their submodules equipped with linear normal mappings. This equivalence reveals a deep structural correspondence between module-theoretic and lattice-theoretic perspectives.
It opens avenues for exploring analogous equivalences in broader algebraic contexts. Future work may investigate how these ideas extend to other classes of modules or to more general types of lattices beyond the semisimple case.

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